plane of the flow, we have from Nochevkina[1] or Grad,[3] for a gas with infinite conductivity,

$$\left.\begin{aligned}&\mathbf{B} = \mathbf{k}\beta\rho(x, y) \qquad b^2 = \frac{B^2}{\mu\rho} = \frac{\beta^2\rho}{\mu} \qquad a^2 = \frac{\gamma p}{\rho}\\ &\frac{q^2}{2} + \frac{a^2}{\gamma - 1} + b^2 = h_0 = \frac{a_0{}^2}{\gamma - 1} + b_0{}^2\\ &K = \left(\frac{\rho_0}{\rho}\right)^2\left[1 - M^2\left(1 + \frac{b^2}{a^2}\right)^{-1}\right] = \\ &\left(\frac{\rho_0}{\rho}\right)^2\left[\frac{\gamma+1}{\gamma-1}a^2 + 3b^2 - 2h_0\right](a^2 + b^2)^{-1}\end{aligned}\right\} \quad (14)$$

Then the vortex flow is given by

$$\psi = \int \frac{\rho}{\rho_0 q}\,dq = -\frac{1}{2\rho_0}\int\frac{(a^2 + b^2)d\rho}{\{h_0 - [a^2/(\gamma - 1)] - b^2\}} \quad (15)$$

whereas the equivalent of the Ringleb solution is

$$\psi = \left[2\left(h_0 - \frac{a^2}{\gamma - 1} - b^2\right)\right]^{-1/2}\sin\theta \quad (16)$$

The corresponding solutions can also be easily obtained in terms of the variable S in the canonical transformation given by Eq. (6). However, it will now be shown that the analytical continuation through the sonic line is better expressed in terms of the Chaplygin transformation σ given by Eq. (5). For an isentropic perfect gas flow we find from Eqs. (5) and (7) that we can write for the Chaplygin transformation

$$\left.\begin{aligned}&\sigma(q) = \int_q^{a_*}\frac{\rho}{\rho_0}\frac{dq}{q} = \sigma'(a_*)(q - a_*) + \frac{\sigma''(a_*)}{2}(q - a_*)^2 + \ldots\\ &\sigma'(a_*) = -\frac{1}{a_*}\left(\frac{\gamma+1}{2}\right)^{-1/(\gamma-1)}\\ &\sigma''(a_*) = \frac{2}{a_*{}^2}\left(\frac{\gamma+1}{2}\right)^{-1/(\gamma-1)}\\ &\frac{q}{a_*} = 1 - \left(\frac{\gamma+1}{2}\right)^{1/(\gamma-1)}\sigma + \left(\frac{\gamma+1}{2}\right)^{2/(\gamma-1)}\sigma^2 + 0(\sigma^3)\end{aligned}\right\} \quad (17)$$

Consequently, any proper analytic function of q can have an analytic continuation across the sonic line. On the other hand, we find from Eqs. (6) and (7) that for the canonical transformation we must write

$$S(q) = \int_q^{a_*}(1 - M^2)^{1/2}\frac{dq}{q} = \left\{\left(\frac{\gamma+1}{\gamma-1}\right)^{1/2}\tanh^{-1}(1 - M^2)^{1/2}\left(\frac{\gamma-1}{\gamma+1}\right)^{1/2} - \tanh^{-1}(1 - M^2)^{1/2}\right\} = -\frac{(1 - M^2)^{3/2}}{3}\left(\frac{2}{\gamma+1}\right) - \frac{\gamma(1 - M^2)^{5/2}}{5}\left(\frac{2}{\gamma+1}\right)^2 + \ldots \quad (18)$$

since $S'(a_*) = 0$ and all the higher derivatives are infinite on the sonic line. Therefore, near the sonic line we have

$$\frac{q}{a_*} = 1 - \left(\frac{1 - M^2}{\gamma + 1}\right) - \left(\frac{2\gamma - 1}{2}\right)\left(\frac{1 - M^2}{\gamma + 1}\right)^2 + 0(1 - M^2)^3 = \left[1 - \left(\frac{3}{2}\right)^{2/3}(\gamma + 1)^{-1/3}(-S)^{2/3} + \left(\frac{5 - 2\gamma}{10}\right)\left(\frac{3}{2}\right)^{4/3}(\gamma + 1)^{-2/3}(-S)^{4/3} + 0(-S)^2\right] \quad (19)$$

Consequently, all of the derivatives of q with respect to the canonical variable S are infinite on the sonic line. It is shown in Ref. 11 how much better the variable σ represents the transonic solution corresponding to the Ringleb[10] flow about a semi-infinite half-plane. Equations (11–13) show how easily series expansions can be obtained by using the variable σ or q itself. These particular series expansions are useful because they immediately give a magnitude estimate, or a correction term, for the effect of a weak magnetic field.

Finally, it should be noted that, although the assumption of an infinite conductivity has some physical significance as a limiting case for an aligned magnetic field, it could only provide a crude approximation to any physically possible flow starting with a transverse magnetic field.

References

1 Nochevkina, I. I., "On the approximation method in investigation of plane rotational flow in magnetohydrodynamics," Soviet Phys. Doklady **4**, 549–533 (1959).

2 Iur'ev, I. M., "On a solution to the equations of magnetogasdynamics," Appl. Math. Mech. (Prikl. Mat. Mekh.) **24**, 233–237 (1960).

3 Grad, H., "Reducible problems in magneto-fluid dynamic steady flows," Rev. Mod. Phys. **32**, 830–847 (1960).

4 Seebas, R., "On transcritical and hypercritical flows in magnetogasdynamics," Quart. Appl. Math. **19**, 231–237 (1961).

5 Tkalich, V. S., "Two-parameter motions in magnetogasdynamics (Gromeka and Chaplygin transformations)," Appl. Math. Mech. (Prikl. Mat. Mekh.) **26**, 130–141 (1962).

6 Smith, P., "The steady magnetodynamic flow of perfectly conducting fluids," J. Math. Mech. **12**, 505–520 (1963).

7 Bers, L. and Gelbart, A., "On a class of differential equations in mechanics of continua," Quart. Appl. Math. **1**, 168–188 (1943).

8 Chaplygin, S., "Gas jets," NACA TM 1063 (1944).

9 Christianovich, S. A., "Flow about a body at high subsonic velocities," Central Aeronautical Institute, Rept. 481 (1940); see NACA TM 1250 (1950).

10 Ringleb, F., "Exakte Lösungen der Differentialgleichungen einer adiabatischen Gasströmung," Z. Angew. Mat. Mekh. **20**, 185–198 (1940).

11 Laitone, E. V., "Exact evaluation of Tricomi's transonic approximation," Phys. Fluids (to be published).

Local Similarity Expansions of the Boundary-Layer Equations

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I. Formulation of the Asymptotic Expansion

CONSIDER the incompressible boundary-layer equation

$$f_{\eta\eta\eta} + ff_{\eta\eta} + \beta(\xi)\{1 - (f_\eta)^2\} = 2\xi\{f_\eta f_{\eta\xi} - f_\xi f_{\eta\eta}\} \quad (1)$$

and its boundary conditions

$$f(\xi, 0) = f_\eta(\xi, 0) = 0 \qquad f_\eta(\xi, \infty) = 1 \quad (2)$$

that have been derived in Ref. 1, for instance. Introducing the inversion wherein the independent variables become (β, η),[1] Eqs. (1) and (2) may be written as

$$f_{\eta\eta\eta} + ff_{\eta\eta} + \beta\{1 - (f_\eta)^2\} = \epsilon(\beta)\{f_\eta f_{\beta\eta} - f_\beta f_{\eta\eta}\} \quad (3)$$

$$f(\beta, 0) = f_\eta(\beta, 0) = 0 \qquad f_\eta(\beta, \infty) = 1 \quad (4)$$

where

$$\epsilon(\beta) = 2\xi\beta'(\xi) = 2\xi(\beta)/\xi'(\beta) \quad (5)$$

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As Merk[1] and Dewey[2] have noted, Eq. (3) shows that the "local similarity" approach will be valid if the right-hand side is everywhere small. Further, it suggests that the appropriate asymptotic expansion for $f(\beta, \eta)$ is

$$f(\beta, \eta) = f_0(\beta, \eta) + \epsilon(\beta)f_1(\beta, \eta) + [\epsilon^2(\beta)/2]f_2(\beta, \eta) + \ldots \quad (6)$$

The derivatives may then be expressed as

$$f_\eta = \{(f_0)_\eta + \epsilon(\beta)(f_1)_\eta + [\epsilon^2(\beta)/2](f_2)_\eta + \ldots\}, \text{etc.} \quad (7)$$

$$f_\beta = \{(f_0)_\beta + \epsilon(\beta)(f_1)_\beta + [\epsilon^2(\beta)/2](f_2)_\beta + \ldots\} + \epsilon'(\beta)\{f_1 + \epsilon(\beta)f_2 + \ldots\}, \text{etc.} \quad (8)$$

However, it should be noted that

$$\epsilon'(\beta) = [1/\beta'(\xi)](d/d\xi)\{2\xi\beta'(\xi)\} \equiv 2\{1 + E(\beta)\} \quad (9a)$$

where

$$E(\beta) = \xi\beta''(\xi)/\beta'(\xi) = -\xi(\beta)\xi''(\beta)/\{\xi'(\beta)\}^2 \quad (9b)$$

Hence, $\epsilon'(\beta) = 0(1)$ if $E(\beta) \leq 0(1)$, a relation that is required if the expansion is to be valid. As a result of Eq. (9), Eq. (8) can be written as

$$f_\beta = [(f_0)_\beta + 2\{1 + E\}f_1] + \epsilon[(f_1)_\beta + 2\{1 + E\}f_2] + \ldots$$
$$f_{\beta\eta} = [(f_0)_{\beta\eta} + 2\{1 + E\}(f_1)_\eta] + \epsilon[(f_1)_{\beta\eta} + 2\{1 + E\}(f_2)_\eta] + \ldots \quad (8')$$

Thus, the substitution of the expansions of Eqs. (6, 7, and 8') into Eq. (3) results in the equation

$$[f_0''' + f_0f_0'' + \beta\{1 - (f_0')^2\}] + \epsilon[f_1''' + f_0f_1'' - 2\beta f_0'f_1' + f_0''f_1] + 0(\epsilon^2) = \epsilon[f_0'\{(f_0')_\beta + 2\{1 + E\}f_1'\} - f_0''\{(f_{0\beta}) + 2\{1 + E\}f_1\}] + 0(\epsilon^2) \quad (10)$$

where the primes denote differentiation with respect to η. Equating powers of ϵ, it follows that the first two terms in the expansion for f, f_0, and f_1 must satisfy the following differential equations and boundary conditions:

$$\left.\begin{array}{l} f_0''' + f_0f_0'' + \beta\{1 - (f_0')^2\} = 0 \\ f_0(\beta, 0) = f_0'(\beta, 0) = 0 \qquad f_0'(\beta, \infty) = 1 \end{array}\right\} \quad (11)$$

$$\left.\begin{array}{l} f_1''' + f_0f_1'' - 2\beta f_0'f_1' + f_0''f_1 + [2\{1 + E\}(f_0''f_1 - f_0'f_1')] = f_0'(f_0')_\beta - f_0''(f_0)_\beta \\ f_1(\beta, 0) = f_1'(\beta, 0) = f_1'(\beta, \infty) = 0 \end{array}\right\} \quad (12)$$

Thus, the terms with the coefficient $2\{1 + E(\beta)\}$ in Eq. (12), which are due to $\epsilon'(\beta)$ being of 0(1), represent contributions to first order in ϵ which do not appear in the analyses of Merk and Dewey.

2. An Approximate Solution for the Shear at the Wall Using Local Similarity Expansions

An approximate solution of the incompressible laminar boundary-layer equation[3] gives the following expression (written in the notation of Sec. 1) for the shear function at the wall:

$$f_{\eta\eta}(\beta, 0) = \frac{1}{a\pi^{1/2}}\left[(2^{1/2} - 1)\left(1 - \frac{\epsilon}{a}\frac{da}{d\beta}\right) + 2^{1/2}\beta\right] \quad (13a)$$

where the function $a(\beta)$ satisfies the equation

$$4a^2 + (\epsilon/a)(da/d\beta) = 1 + [1 + (2/\pi)]\beta \quad (13b)$$

Table 1 Shear functions at wall

β	τ_0	τ_1
−0.1	0.337	0.409
0	0.467	0.204
0.1	0.581	0.119
0.2	0.683	0.076
0.3	0.775	0.057
0.4	0.860	0.037
0.5	0.938	0.027
0.6	1.012	0.021
0.8	1.148	0.013
1.0	1.271	0.008

Since it has been shown that, for strict similarity $[(\epsilon/a) \times (da/d\beta) \equiv 0]$, this expression differs very little from the exact result in the range $-0.1 \leq \beta \leq 1.0$; this equation is studied in this section to get an idea of the correction terms in the case of local similarity ($\epsilon \to 0$).

To solve Eq. (13) for this case, consider the local similarity expansion of $a(\beta)$:

$$a(\beta) = a_0 + \epsilon a_1 + (\epsilon^2/2)a_2 + \ldots \quad (14a)$$

with

$$a^2 = a_0^2 + 2\epsilon a_0a_1 + \ldots$$

$$\frac{da}{d\beta} = \left[\frac{da_0}{d\beta} + 2(1 + E)a_1\right] + \epsilon\left[\frac{da_1}{d\beta} + 2(1 + E)a_2\right] + \ldots \quad (14b)$$

Substitution of these expansions into Eq. (13b) gives

$$4a_0^2 + \epsilon\left\{\frac{1}{a_0}\left[\frac{da_0}{d\beta} + 2(1 + E)a_1\right] + 8a_0a_1\right\} + \ldots = 1 + \left(1 + \frac{2}{\pi}\right)\beta \quad (15)$$

Therefore, the leading terms in the expansion for $a(\beta)$, a_0 and a_1, are

$$a_0 = \tfrac{1}{2}\{1 + [1 + (2/\pi)]\beta\}^{1/2} \quad (16a)$$

$$a_1 = \frac{-[1 + (2/\pi)]}{8\{1 + [1 + (2/\pi)]\beta\}^{1/2}\{2 + [1 + (2/\pi)]\beta + E\}} \quad (16b)$$

This means that the shear function at the wall is

$$f_{\eta\eta}(\beta, 0) = f_0''(\beta, 0) + \epsilon f_1''(\beta, 0) + \ldots = \tau_0(\beta)[1 - \epsilon\tau_1(\beta) + \ldots] \quad (17)$$

where

$$\tau_0(\beta) = \frac{2[(2^{1/2} - 1) + 2^{1/2}\beta]}{\pi^{1/2}\{1 + [1 + (2/\pi)]\beta\}^{1/2}} \quad (18a)$$

$$\tau_1(\beta) = \frac{(2^{1/2} - 1)[1 + (2/\pi)]\{1 - [2^{1/2} - (4/\pi)]\beta\}}{4[(2^{1/2} - 1) + 2^{1/2}\beta]\{1 + [1 + (2/\pi)]\beta\}\{2 + [1 + (2/\pi)]\beta + E\}} \quad (18b)$$

Table 1 gives the values of $\tau_0(\beta)$ and $\tau_1(\beta)$ for $E = 0$. From this table, it can be seen that the correction term τ_1 is most important for small values of β and becomes less important as β increases. For $E = 0(1)$, τ_1 is yet smaller.

References

[1] Merk, H. J., "Rapid calculations for boundary-layer transfer using wedge solutions and asymptotic expansions," J. Fluid Mech. 5, 460–480 (1959).

[2] Dewey, C. F., "Use of local similarity concepts in hypersonic viscous interaction problems," AIAA J. 1, 20–33 (1963).

[3] Bush, W. B., "A method of obtaining an approximate solution of the laminar boundary-layer equations," J. Aerospace Sci. 28, 350–351 (1961).